On the KdV/KP-I regime for the Nonlinear Schrödinger equation

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Setting

We consider the non linear Schrödinger equation in $\mathbb{R}^d$

$$i\frac{\partial \Psi}{\partial \tau} + \Delta \Psi = \Psi f(|\Psi|^2) \qquad (NLS)$$

$$\Psi(\tau, x) \in \mathbb{C}, \qquad \tau \geq 0, \qquad x \in \mathbb{R}^d.$$

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We assume

$$f(1) = 0,$$

example: $f(r) = r - 1 \quad \longrightarrow$ NLS=GP

$\Psi = 1$ is a trivial solution of (NLS). We consider an initial datum

$$\Psi_{|\tau=0} \simeq 1.$$

Hydrodynamic form of (NLS)

Madelung's transform: (for $|\Psi| > 0$)

$$\Psi = a \exp(i\phi),$$

with

$$a(\tau, x) \in \mathbb{R}^+, \qquad \phi(\tau, x) \in \mathbb{R}, \qquad v \equiv \nabla\phi.$$

$$(NLS) \qquad \Longleftrightarrow \qquad \begin{cases} \partial_\tau(a^2) + \nabla \cdot \left(a^2 v\right) = 0 \\ \\ \partial_\tau v + 2\left(v \cdot \nabla\right) v + \nabla\left(f(a^2)\right) = \nabla\left(\dfrac{\Delta a}{a}\right). \end{cases}$$

Compressible Euler type system plus *quantum pressure.*

(KdV) and (KP-I) Equations

⋆ Two equations describing unidirectional waves :

$$2\partial_t \zeta + \Gamma\, \zeta \partial_z \zeta - \frac{1}{\mathfrak{c}_s^2}\, \partial_z^3 \zeta = 0, \qquad (d = 1) \qquad \text{(KdV)}$$

and

$$2\partial_t \zeta + \Gamma\, \zeta \partial_{z_1} \zeta - \frac{1}{\mathfrak{c}_s^2}\, \partial_{z_1}^3 \zeta + \Delta_{z_\perp} \partial_{z_1}^{-1} \zeta = 0 \qquad (d \geq 2). \qquad \text{(KP-I)}$$

where $(f'(1) > 0)$

$$\mathfrak{c}_s \equiv \sqrt{f'(1)} > 0 \qquad \text{and} \qquad \Gamma \equiv 6 + \frac{4}{\mathfrak{c}_s^2} f''(1) \in \mathbb{R}.$$

Formal derivation of (KdV) and (KP-I)

Ansatz $\Psi(\tau, x) = (1 + \varepsilon^2 A^\varepsilon(t, z)) \exp(i\varepsilon\varphi^\varepsilon(t, z))$ with

$$t = \mathfrak{c}_s \varepsilon^3 \tau, \quad z_1 = \varepsilon(x_1 - \mathfrak{c}_s t), \quad z_\perp = \varepsilon^2 x_\perp.$$

$$(NLS) \qquad \Longleftrightarrow \qquad \begin{cases} \partial_\tau(a^2) + \nabla \cdot \big(a^2 v\big) = 0 \\ \partial_\tau v + 2\big(v \cdot \nabla\big) v + \nabla\big(f(a^2)\big) = \nabla\left(\dfrac{\Delta a}{a}\right). \end{cases}$$

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$$t = \mathfrak{c}_s \varepsilon^3 \tau, \quad z_1 = \varepsilon(x_1 - \mathfrak{c}_s t), \quad z_\perp = \varepsilon^2 x_\perp.$$

$$u^\varepsilon \equiv \frac{1}{\mathfrak{c}_s} \partial_{z_1} \varphi^\varepsilon.$$

$$(d = 1) \qquad \begin{cases} \partial_t A^\varepsilon - \frac{1}{\varepsilon^2} \partial_z A^\varepsilon + 2u^\varepsilon \partial_z A^\varepsilon + \frac{1}{\varepsilon^2} \partial_z u^\varepsilon + A^\varepsilon \partial_z u^\varepsilon = 0 \\ \partial_t u^\varepsilon - \frac{1}{\varepsilon^2} \partial_z u^\varepsilon + 2u^\varepsilon \partial_z u^\varepsilon + \frac{1}{\varepsilon^2} \partial_z A^\varepsilon + (\Gamma - 5) A^\varepsilon \partial_z A^\varepsilon \\ \qquad = \partial_z \Big(\frac{\partial_z^2 A^\varepsilon}{\mathfrak{c}_s^2 (1 + \varepsilon^2 A^\varepsilon)} \Big) + \varepsilon^2 \partial_z \big(\mathcal{O}([A^\varepsilon]^3) \big). \end{cases}$$

Formal derivation of (KdV) and (KP-I)

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If A^ε and u^ε are indeed of order one, we must have

$$A^\varepsilon \simeq u^\varepsilon = \frac{1}{\mathfrak{c}_s}\partial_z \varphi^\varepsilon \to A.$$

Formal derivation of (KdV) and (KP-I)

Summing the two equations :

$$\Longrightarrow \qquad \partial_t\big(A^\varepsilon + u^\varepsilon\big) + 2u^\varepsilon\partial_z A^\varepsilon + A^\varepsilon\partial_z u^\varepsilon + 2u^\varepsilon\partial_z u^\varepsilon + (\Gamma - 5)A^\varepsilon\partial_z A^\varepsilon$$

$$= \partial_z\Big(\frac{\partial_z^2 A^\varepsilon}{\mathfrak{c}_s^2(1+\varepsilon^2 A^\varepsilon)}\Big) + \mathcal{O}(\varepsilon^2)$$

$$\Downarrow$$

$$2\partial_t A + \Gamma A\partial_z A - \frac{1}{\mathfrak{c}_s^2}\partial_z^3 A = 0.$$

Formal derivation of (KdV) and (KP-I)

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$$2\partial_t A + \Gamma A\partial_z A - \frac{1}{\mathfrak{c}_s^2}\partial_z^3 A = 0.$$

◇ Derivation standard in the physical literature :
Condensats: Jones-Roberts (GP, $d = 2$ ou 3)
Nonlinear Optics : Kivshar (general f, $d = 1$, $d = 2$).

Convergence results towards KdV/KP-I

Theorem 1 [C.-Rousset] ($d = 1$): We assume

$$\sup_{0<\varepsilon<1} \Big\{ \|A_0^\varepsilon\|_{H^1} + \frac{1}{\varepsilon} \Big\| \frac{\partial_z \varphi_0^\varepsilon}{\mathfrak{c}_s} - A_0^\varepsilon \Big\|_{L^2} \Big\} < +\infty \qquad \text{and} \qquad A_0^\varepsilon \to A_0 \quad \text{in} \quad L^2.$$

then, the solution Ψ of (NLS) with $\Psi_{|\tau=0} = (1 + \varepsilon^2 A_0^\varepsilon) \exp\big(i\varepsilon\varphi_0^\varepsilon\big)$ writes $\Psi(\tau, x) = (1 + \varepsilon^2 A^\varepsilon(t,z)) \exp\big(i\varepsilon\varphi^\varepsilon(t,z)\big)$ with

$$\sup_{\varepsilon,\ t\geq 0} \Big\{ \|A^\varepsilon\|_{H^1} + \frac{1}{\varepsilon} \Big\| \frac{\partial_z \varphi^\varepsilon}{\mathfrak{c}_s} - A^\varepsilon \Big\|_{L^2} \Big\} < +\infty.$$

If A is the H^1 solution of KdV with $A_{|t=0} = A_0$, then, as $\varepsilon \to 0$,

$$\begin{cases} A^\varepsilon \to A & \text{in} \quad \mathcal{C}^0_{loc}(\mathbb{R}_+, H^{1^-}) \\ \partial_z \varphi^\varepsilon / \mathfrak{c}_s \to A & \text{in} \quad \mathcal{C}^0_{loc}(\mathbb{R}_+, L^2). \end{cases}$$

Convergence results towards KdV/KP-I

In the very well prepared case

$$\left\|\partial_z \varphi_0^\varepsilon - \mathfrak{c}_s A_0^\varepsilon\right\|_{L^2} = o(\varepsilon) \qquad \text{and} \qquad A_0^\varepsilon \longrightarrow A_0 \qquad \text{in} \quad H^1,$$

we have convergence up to H^1:

$$A^\varepsilon \longrightarrow A \qquad \text{in} \quad \mathcal{C}^0_{loc}\big(\mathbb{R}_+, H^1\big)$$

and

$$\partial_z \varphi^\varepsilon - \mathfrak{c}_s A^\varepsilon = o(\varepsilon) \qquad \text{in} \quad \mathcal{C}^0_{loc}\big(\mathbb{R}_+, L^2\big).$$

Convergence results towards KdV/KP-I

Theorem 2 [C.-Rousset] ($d \geq 1$): For $s > 1 + \frac{d}{2}$,we suppose

$$\sup_{0<\varepsilon<1} \left\| \left(A_0^\varepsilon, \partial_{z_1}\varphi_0^\varepsilon, \varepsilon\nabla_{z_\perp}\varphi_0^\varepsilon \right) \right\|_{H^{s+1}} < +\infty \qquad \text{and} \qquad A_0^\varepsilon \to A_0 \quad \text{dans} \quad L^2$$

as well as

$$\begin{cases} (d=1) & \left\| \mathfrak{c}_s A_0^\varepsilon - \partial_{z_1}\varphi_0^\varepsilon \right\|_{L^2} \to 0 \\ (d \geq 2) & \left\| \mathfrak{c}_s A_0^\varepsilon - \partial_{z_1}\varphi_0^\varepsilon \right\|_{L^2} = \mathcal{O}(\varepsilon) \end{cases} \qquad \text{as} \quad \varepsilon \to 0.$$

Then, for some $T > 0$, the solution Ψ of (NLS)s writes
$\Psi(\tau, x) = (1 + \varepsilon^2 A^\varepsilon(t,z)) \exp\left(i\varepsilon\varphi^\varepsilon(t,z)\right)$ for $0 \leq t \leq T$ with

$$\sup_{\varepsilon,\ 0\leq t\leq T} \left\{ \left\| A^\varepsilon \right\|_{H^{s+1}} + \left\| \left(\partial_{z_1}\varphi^\varepsilon, \varepsilon\nabla_{z_\perp}\varphi^\varepsilon \right) \right\|_{H^s} \right\} < +\infty.$$

Convergence results towards (KdV)/(KP-I)

If $A \in L^\infty([0,T], H^s)$ is the solution to (KdV)/(KP-I) with $A_{|t=0} = A$, then

$$\begin{cases} A^\varepsilon \to A & \text{in} \quad \mathcal{C}\big([0,T], H^{s+1^-}\big) \\ \dfrac{\partial_{z_1}\varphi^\varepsilon}{\mathfrak{c}_s} \to A & \text{in} \quad \mathcal{C}\big([0,T], H^{s^-}\big). \end{cases}$$

Convergence results towards (KdV)/(KP-I)

In the non well-prepared case

$$A_0^\varepsilon \to A_0 \qquad \partial_{z_1}\varphi^\varepsilon/\mathfrak{c}_s \to (u_0)_1 \qquad \text{and} \qquad \nabla_{z_\perp}\varphi^\varepsilon/\mathfrak{c}_s \to 0 \qquad \text{in} \quad L^2,$$

we show

$$\frac{1}{2}\Big(A^\varepsilon + \partial_{z_1}\varphi^\varepsilon/\mathfrak{c}_s\Big) \to A \qquad \text{in} \quad L^2\big([0,T], H^{s^-}\big)$$

and

$$A^\varepsilon,\ \partial_{z_1}\varphi^\varepsilon/\mathfrak{c}_s \rightharpoonup A \qquad \text{weakly in} \quad L^2\big([0,T]\times\mathbb{R}^d\big),$$

where A is the solution to (KdV)/(KP-I) with $A_{|t=0} = \frac{1}{2}\Big(A_0 + (u_0)_1\Big)$.

Use of the complete integrability of (KdV) and (NLS)

Work of F. Béthuel - P. Gravejat - J-C. Saut - D. Smets:

$\rightarrow$ Convergence towards (KdV) for $f(r) = r - 1$ (integrable case) using the high order conservation laws.

Theorem [Béthuel-Gravejat-Saut-Smets]: We assume

$$\|A^\varepsilon\|_{H^3} + \|\partial_z \varphi^\varepsilon / \mathfrak{c}_s\|_{H^3} \leq M$$

and we consider the solution ζ^ε of (KdV) with

$$\zeta^\varepsilon_{|t=0} = A_0^\varepsilon.$$

Then, for $0 < \varepsilon < \varepsilon_0(M)$

$$\|A^\varepsilon - \zeta^\varepsilon\|_{L^2} \leq C_M \Big(\|A_0^\varepsilon - \partial_z \varphi_0^\varepsilon / \mathfrak{c}_s\|_{H^3} + \varepsilon^2 \Big) \mathrm{e}^{C_M t}.$$

Transonic limit for the travelling waves

Two quantities (formally) conserved by the Schrödinger flow

$$i\frac{\partial \Psi}{\partial \tau} + \Delta \Psi = \Psi f(|\Psi|^2) \qquad (NLS)$$

the *energy* ($F' = f$, $F(r_0^2) = 0$)

$$\mathcal{E} \equiv \frac{1}{2}\int_{\mathbb{R}^d} |\nabla \Psi|^2 + F(|\Psi|^2)\, dx$$

and the *momentun* (with $\Psi = \rho \mathrm{e}^{i\phi}$)

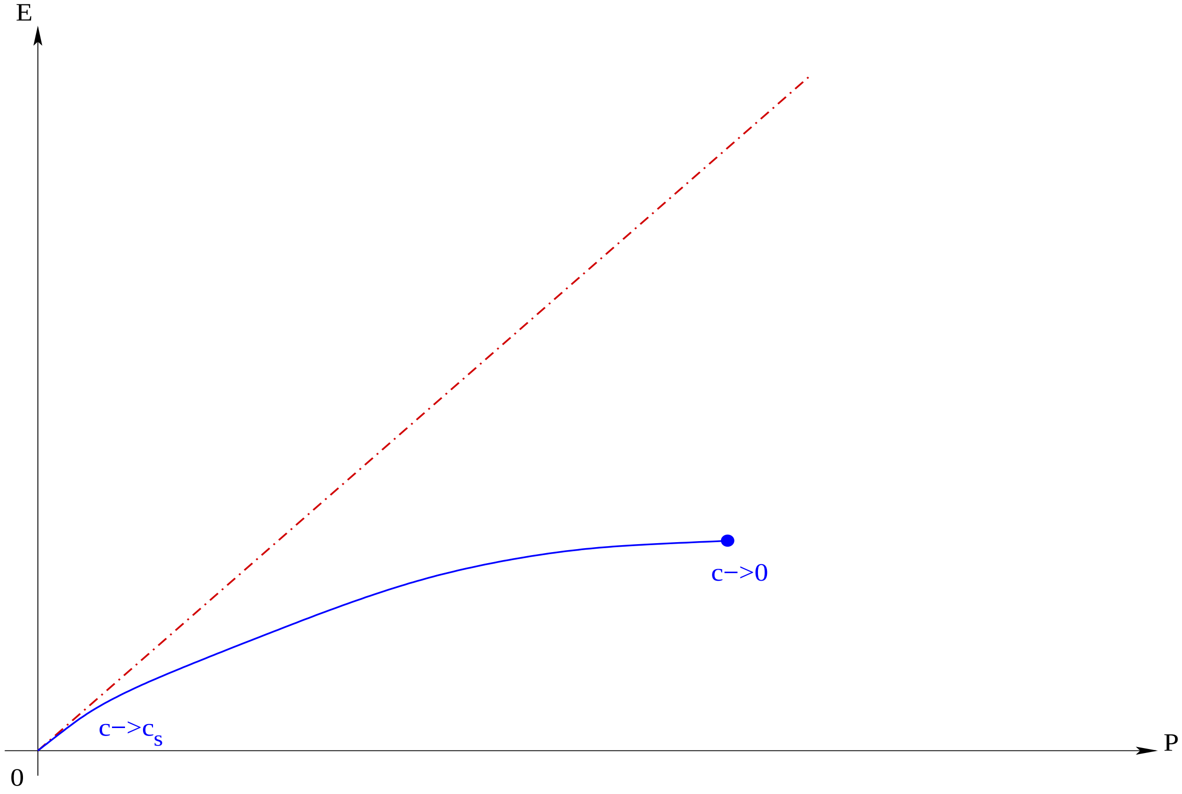
$$\vec{\mathcal{P}} \equiv \frac{1}{2}\int_{\mathbb{R}^d} {''}\rho^2 \nabla \phi{''}\, dx = \frac{1}{2}\int_{\mathbb{R}^d} {''}(\rho^2 - 1)\nabla \phi{''}\, dx.$$

Travelling waves : $\Psi(\tau, x) = U(x_1 - c\tau, x_\perp)$ solve

$$\Delta U = U f(|U|^2) - ic\partial_{x_1} U.$$

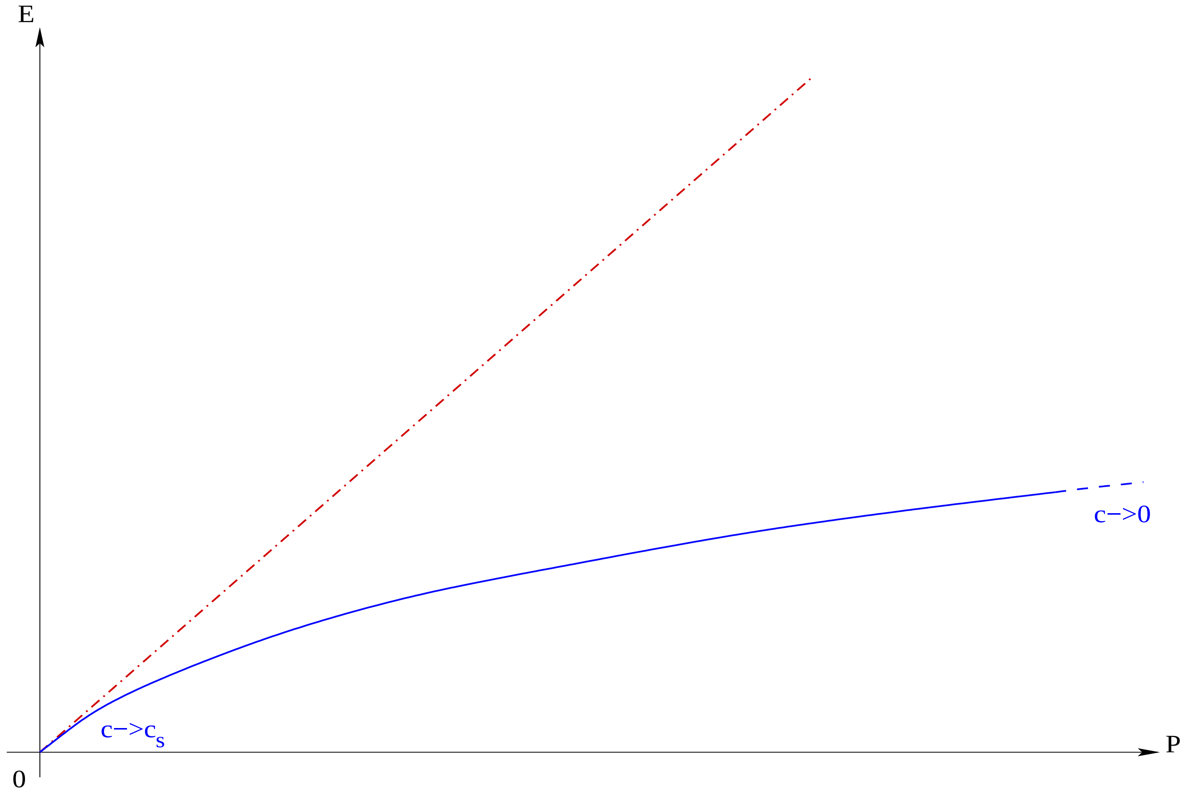
Transonic limit for the travelling waves

(E, P) diagram for the travelling waves in dimension $d = 1$, with $f(r) = r - 1$.



Transonic limit for the travelling waves

(E, P) diagram for the travelling waves in dimension $d = 2$, with $f(r) = r - 1$ (Jones-Roberts (1982)).



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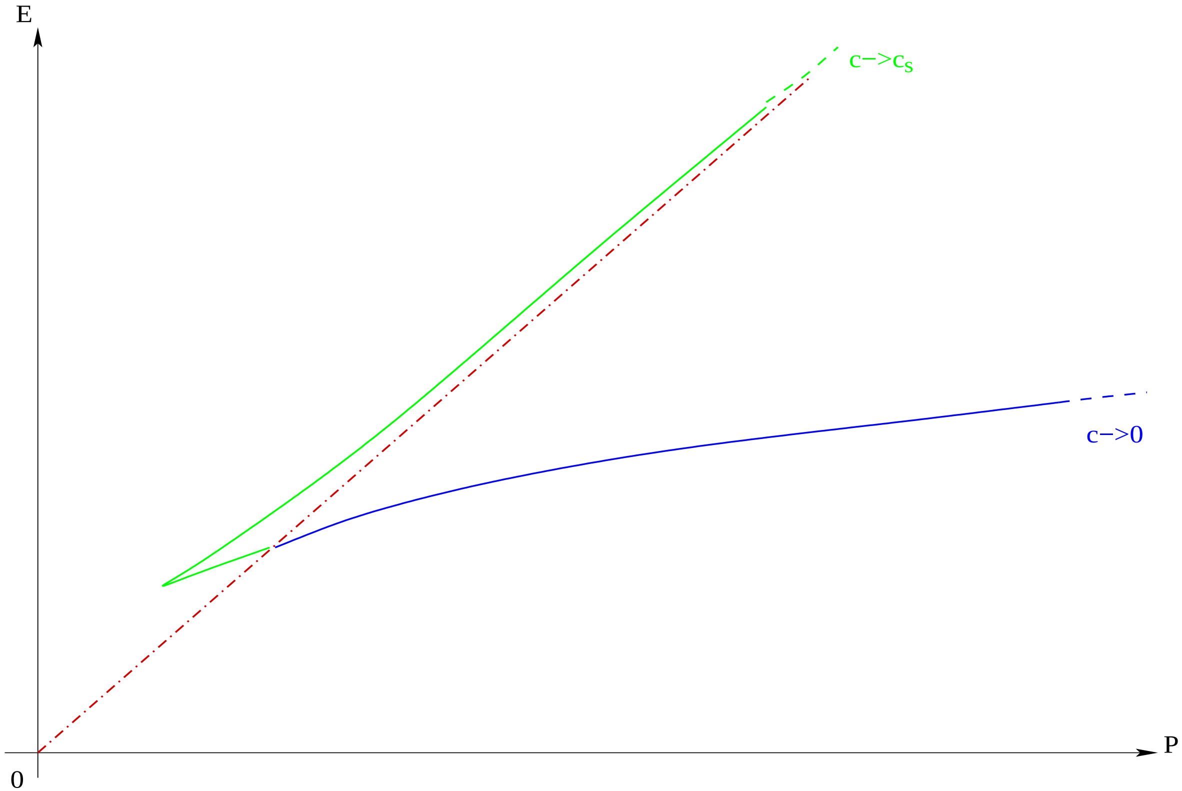

Transonic limit for the travelling waves

Theorem [Béthuel-Gravejat-Saut]: We assume $d = 2$ or $d = 3$ and $f(r) = r - 1$. There exist at least one minimizer of the energy under constraint of fixed momentum $\vec{\mathcal{P}}_1 = \mathfrak{p}$

- for any $\mathfrak{p} > 0$ if $d = 2$;
- for any $\mathfrak{p} \geq \mathfrak{p}_0 > 0$ if $d = 3$.

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Theorem [Mariş]: We assume $d \geq 3$ and $f'(1) > 0$. Then, for every $0 < c < \mathfrak{c}_s$, there exists at least one nontrivial travelling wave for (NLS) with speed c.

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Theorem [C.-Mariş]: We assume $d = 2$, $f'(1) > 0$ and $\Gamma \neq 0$. Then, there exists a set $\mathcal{C} \subset (0, \mathfrak{c}_s)$ of speeds such that for any $c \in \mathcal{C}$, there exists at least one nontrivial travelling wave for (NLS) with speed c. Moreover, $\sup \mathcal{C} = \mathfrak{c}_s$ and $\inf \mathcal{C} = 0$.

Transonic limit for the travelling waves

Ansatz $U(x) = (1 + \varepsilon^2 A^\varepsilon(z)) \exp(i\varepsilon\varphi^\varepsilon(z))$ with

$$z_1 = \varepsilon x_1, \qquad z_\perp = \varepsilon^2 x_\perp, \qquad c = c(\varepsilon) = \sqrt{\mathfrak{c}_s^2 - \varepsilon^2}.$$

We expect $A^\varepsilon \simeq \partial_{z_1}\varphi^\varepsilon / \mathfrak{c}_s \to \mathcal{W}$ solitary wave of (KdV)/(KP-I):

$$\frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\mathcal{W} + \Gamma\mathcal{W}\partial_{z_1}\mathcal{W} - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3\mathcal{W} + \Delta_{z_\perp}\partial_{z_1}^{-1}\mathcal{W} = 0.$$

Works of de Bouard-Saut: if $\Gamma \neq 0$, existence of ground states with speed $1/\mathfrak{c}_s^2$ only for $d = 2$ and $d = 3$.

If $d = 1$, “the“ (KdV) soliton (KdV) is

$$\mathrm{w}(z) \equiv -\frac{6}{\mathfrak{c}_s^2 \Gamma \mathrm{ch}^2(z/2)},$$

Transonic limit for the travelling waves

Theorem [C.-Mariş]: We assume $\Gamma \neq 0$, $d = 1$, 2 or 3 and $f'(1) > 0$. There exists a sequence (U_n, c_n), where U_n is a non trivial travelling wave with speed c_n and $c_n = \sqrt{\mathfrak{c}_s^2 - \varepsilon_n^2} \to \mathfrak{c}_s$ such that

$$U_n(x) = (1 + \varepsilon_n^2 A_n(z)) \exp(i\varepsilon_n \varphi_n(z)) \qquad z_1 = \varepsilon_n x_1, \qquad z_\perp = \varepsilon_n^2 x_\perp.$$

Moreover, there exists a ground state $\mathcal{W}$ of (KdV)/(KP-I) such that

$$A_n \to \mathcal{W} \qquad \text{and} \qquad \partial_{z_1} \varphi_n / \mathfrak{c}_s \to \mathcal{W}$$

in all the $W^{s,p}$, $s \in \mathbb{N}$, $1 < p \leq \infty$. Finally,

$$E(U_n) \sim \mathfrak{c}_s P(U_n) \sim K \varepsilon_n^{5-2d}.$$

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◇ If $d = 1$, it is a branch of travelling waves for c near $\mathfrak{c}_s$.

Transonic limit for the travelling waves

Generalize similar result of Béthuel-Gravejat-Saut for $d = 2$ and $f(r) = r - 1$.

Difficulties :

- minimisation under constraint is impossible :

  $*$ F can take negative values (cubic-quintic nonlinearity);
  $*$ in dim $d = 3$, the travelling waves are not minimizers under constraint.

- $d = 3$: high energy/momentum.

- do the solution vanish ? $\Rightarrow$ $c \simeq \mathfrak{c}_s$ and $E(u) - c\vec{\mathcal{P}}_1(u)$ small is sufficient.

- getting the L^p bounds in dim $d = 3$ much longer (Sobolev...).

What happens when $\Gamma = 0$?

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Principe : expand to next order the nonlinearity and enlarge A^ε and φ^ε.

⇒ we shall obtain a *cubic* (KdV)/(KP-I) equation :

$$\frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1}\mathcal{W} - \frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1}^3\mathcal{W} + \Gamma'\mathcal{W}^2\partial_{z_1}\mathcal{W} + \Delta_{z_\perp}\partial_{z_1}^{-1}\mathcal{W} = 0,$$

where the coefficient Γ' is given by

$$\Gamma' \equiv \frac{1}{\mathfrak{c}_s^2}\Big(6f''(1) + 4f'''(1)\Big) - 15 = \frac{4f'''(1)}{\mathfrak{c}_s^2} - 24$$

(recall $2f''(1) = -3\mathfrak{c}_s^2$ since $\Gamma = 0$).

⇒ similar result for planar ferromagnets (Spathis-Papanicolaou).

Formal derivation of cubic (KdV)/(KP-I) ($\Gamma = 0$)

We plug the ansatz ($c(\varepsilon) = \sqrt{\mathfrak{c}_s^2 - \varepsilon^2}$)

$$U(x) = r_0\Big(1 + \varepsilon A_\varepsilon(z)\Big)\exp\big(i\varphi_\varepsilon(z)\big) \qquad z_1 \equiv \varepsilon x_1, \quad z_\perp \equiv \varepsilon^2 x_\perp$$

$$\begin{cases} -c(\varepsilon)\partial_{z_1}A_\varepsilon + 2\varepsilon\partial_{z_1}\varphi_\varepsilon\partial_{z_1}A_\varepsilon + 2\varepsilon^3\nabla_{z_\perp}\varphi_\varepsilon \cdot \nabla_{z_\perp}A_\varepsilon \\ \qquad + (1+\varepsilon A_\varepsilon)\Big(\partial_{z_1}^2\varphi_\varepsilon + \varepsilon^2\Delta_{z_\perp}\varphi_\varepsilon\Big) = 0 \\ \\ -c(\varepsilon)\partial_{z_1}\varphi_\varepsilon + \varepsilon(\partial_{z_1}\varphi_\varepsilon)^2 + \varepsilon^3|\nabla_{z_\perp}\varphi_\varepsilon|^2 + \dfrac{1}{\varepsilon}f\Big(r_0^2(1+\varepsilon A_\varepsilon)^2\Big) \\ \qquad -\varepsilon^2\dfrac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon A_\varepsilon} = 0. \end{cases}$$

Formal derivation of cubic (KdV)/(KP-I) ($\Gamma = 0$)

To leading order :

$$\partial_{z_1} \varphi_\varepsilon \simeq \mathfrak{c}_s A_\varepsilon.$$

To second order :

$$\partial_{z_1} \varphi_\varepsilon - c(\varepsilon) A_\varepsilon = -\frac{3\varepsilon}{2} \mathfrak{c}_s A_\varepsilon^2 + o(\varepsilon).$$

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Combining $(1)c(\varepsilon)/\mathfrak{c}_s^2 + \partial_{z_1}(2)/\mathfrak{c}_s^2$ and dividing by ε^2:

$$\frac{1}{\mathfrak{c}_s^2}\partial_{z_1}A_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\Big(\frac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon A_\varepsilon}\Big) + \frac{c(\varepsilon)}{\mathfrak{c}_s^2}(1+\varepsilon^2 A_\varepsilon)\Delta_{z_\perp}\varphi_\varepsilon$$
$$+ \frac{1}{\varepsilon}\Big\{2\frac{c(\varepsilon)}{\mathfrak{c}_s^2}\partial_{z_1}\varphi_\varepsilon\partial_{z_1}A_\varepsilon + \frac{c(\varepsilon)}{\mathfrak{c}_s^2}A_\varepsilon\partial_{z_1}^2\varphi_\varepsilon + \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}[(\partial_{z_1}\varphi_\varepsilon)^2] - 5A_\varepsilon\partial_{z_1}A_\varepsilon\Big\}$$
$$+ \frac{1}{\mathfrak{c}_s^2}\Big(6f''(1) + 4f'''(1)\Big)A_\varepsilon^2\partial_{z_1}A_\varepsilon = \mathcal{O}(\varepsilon).$$

Formal derivation of cubic (KdV)/(KP-I) ($\Gamma = 0$)

$$\Longrightarrow \qquad \frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1} A + \Gamma' A^2 \partial_{z_1} A - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A + \Delta_{z_\perp}\partial_{z_1}^{-1} A = 0$$

Existence of travelling waves ?

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Existence of travelling waves ?

Basic remarks :

- if A is a solution, so is $-A$!

- for $\Gamma' > 0$: defocusing equation / for $\Gamma' < 0$: focusing equation.

Formal derivation of cubic (KdV)/(KP-I) ($\Gamma = 0$)

$$\Longrightarrow \qquad \frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1} A + \Gamma' A^2 \partial_{z_1} A - \frac{1}{\mathfrak{c}_s^2} \partial_{z_1}^3 A + \Delta_{z_\perp} \partial_{z_1}^{-1} A = 0$$

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Basic remarks :

- if A is a solution, so is $-A$!

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When $d = 1$, $\Gamma' < 0$, “the“ (KdV’) soliton is

$$\mathrm{w}^{\pm}(z) \equiv \pm \frac{\sqrt{-6/(\Gamma' \mathfrak{c}_s^2)}}{\mathrm{ch}(z)}.$$

Works of de Bouard-Saut: for $\Gamma' < 0$, existence of ground states (with speed $1/(2\mathfrak{c}_s^2)$) only for $d = 2$.

Formal derivation of cubic (KdV)/(KP-I) ($\Gamma = 0$)

Théorème [C.-Mariş]: We assume $f'(1) > 0$, $\Gamma = 0 > \Gamma'$ and $d = 1$. There exists $0 < \mathfrak{c}_0 < \mathfrak{c}_s$ s.t. for every $\mathfrak{c}_0 < c = c(\varepsilon) < \mathfrak{c}_s$, there exists exactly two travelling waves $U_\varepsilon^\pm$ with speed $c(\varepsilon)$. Moreover,

$$U_\varepsilon^\pm(x) = (1 + \varepsilon A_\varepsilon^\pm(z)) \exp(i\varphi_\varepsilon^\pm(z)), \qquad z = \varepsilon x,$$

with

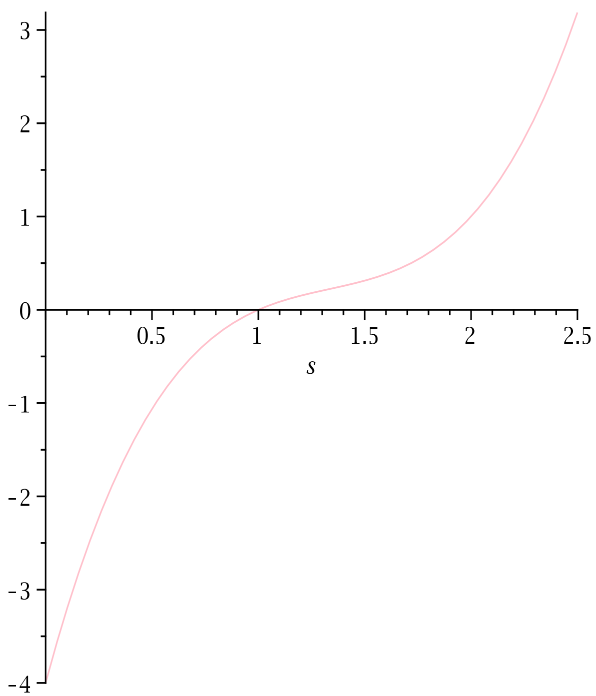
$$A_\varepsilon^\pm \to \mathrm{w}^\pm \qquad \text{and} \qquad \partial_z \varphi_\varepsilon^\pm / \mathfrak{c}_s \to \mathrm{w}^\pm$$

in all the $W^{s,p}$, $s \in \mathbb{N}$, $1 < p \leq \infty$.

Other Energy/Momentum diagram in dimension $d = 1$

Actually, when $d = 1$, the Energy/Momentum diagrams can be curious

Ex1 : $$f(r) = r - 1 - \frac{3}{2}(r-1)^2 + \frac{3}{2}(r-1)^3 \qquad (\Gamma = 0, \quad \Gamma' = -6)$$



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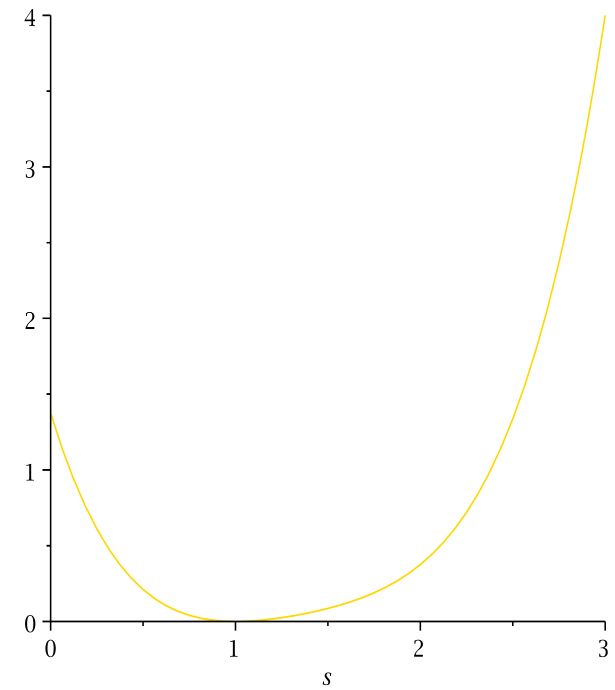

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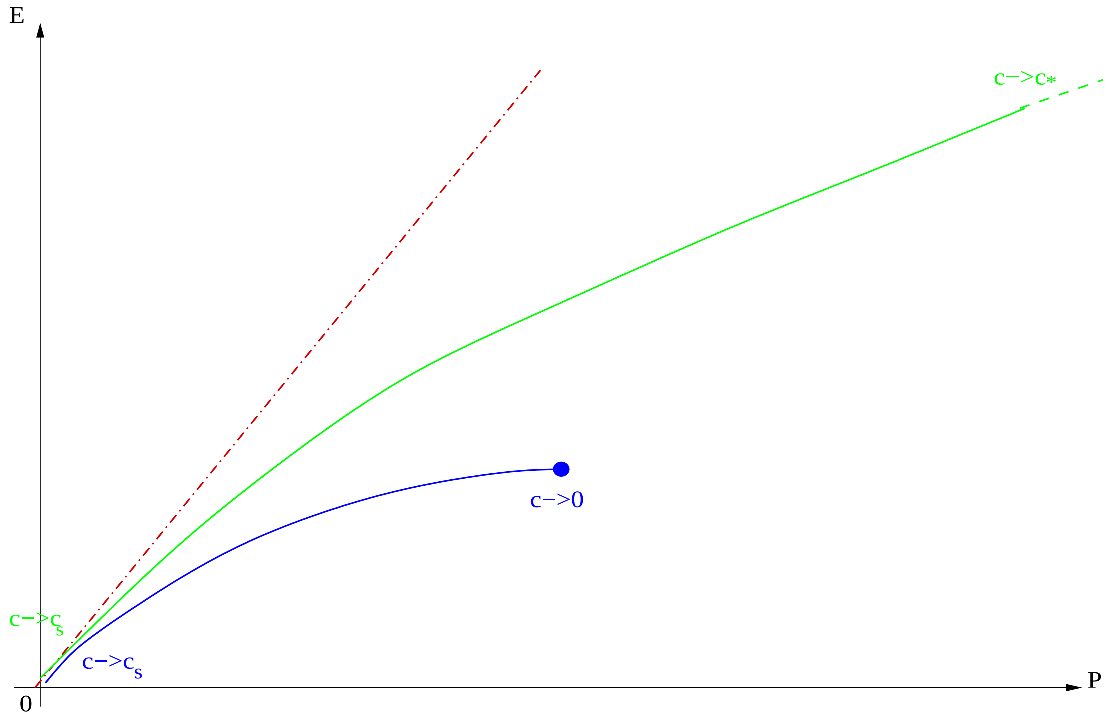

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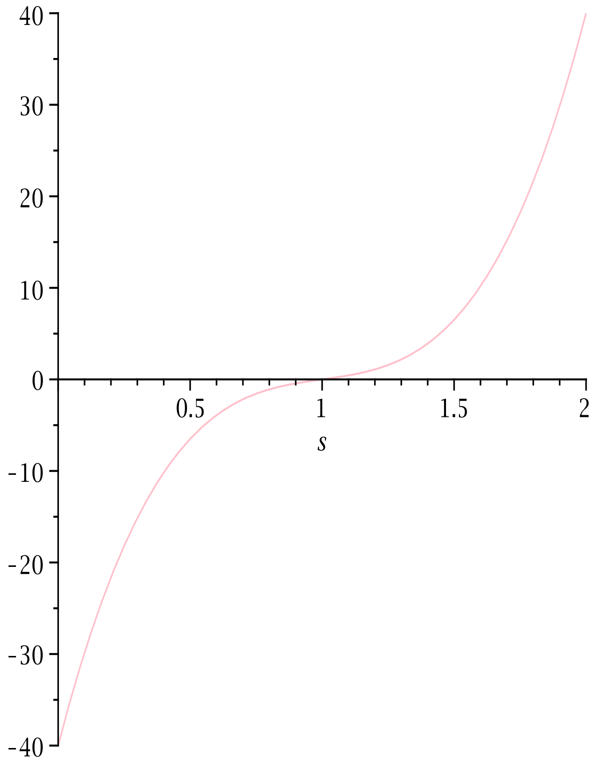
Ex2 : $$f(r) = 4(r-1) + 36(r-1)^3 \qquad (\Gamma = 6)$$



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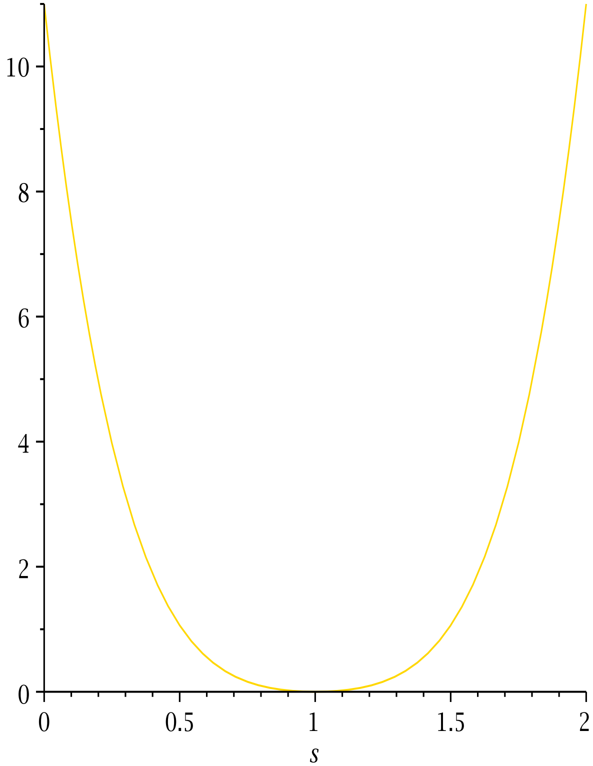

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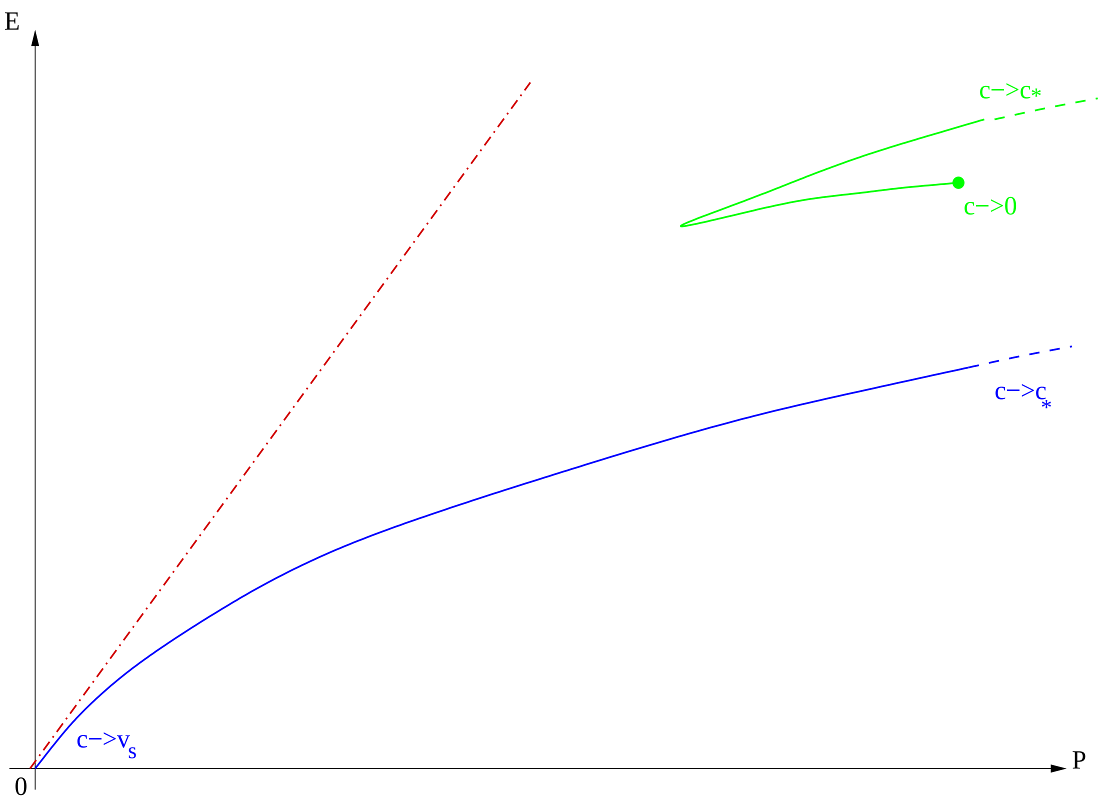

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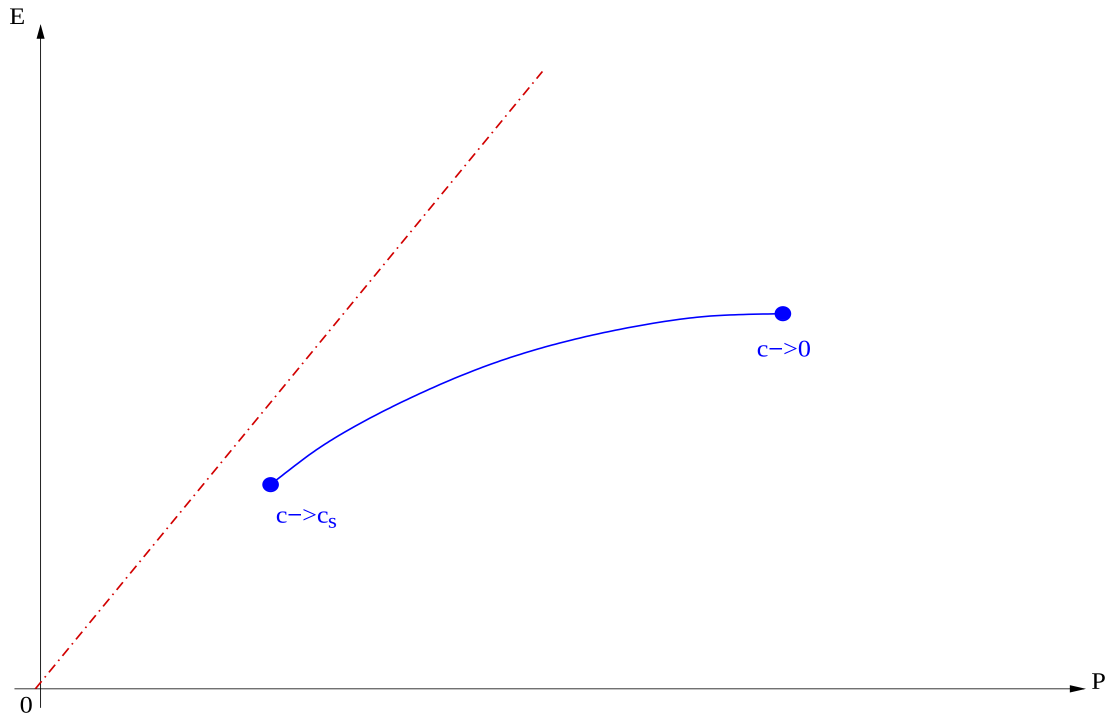
Ex3 : $$f(r) = \frac{1}{2}(r-1) - \frac{3}{4}(r-1)^2 + 2(r-1)^3 \qquad (\Gamma = 0,\ \Gamma' = 24 > 0)$$



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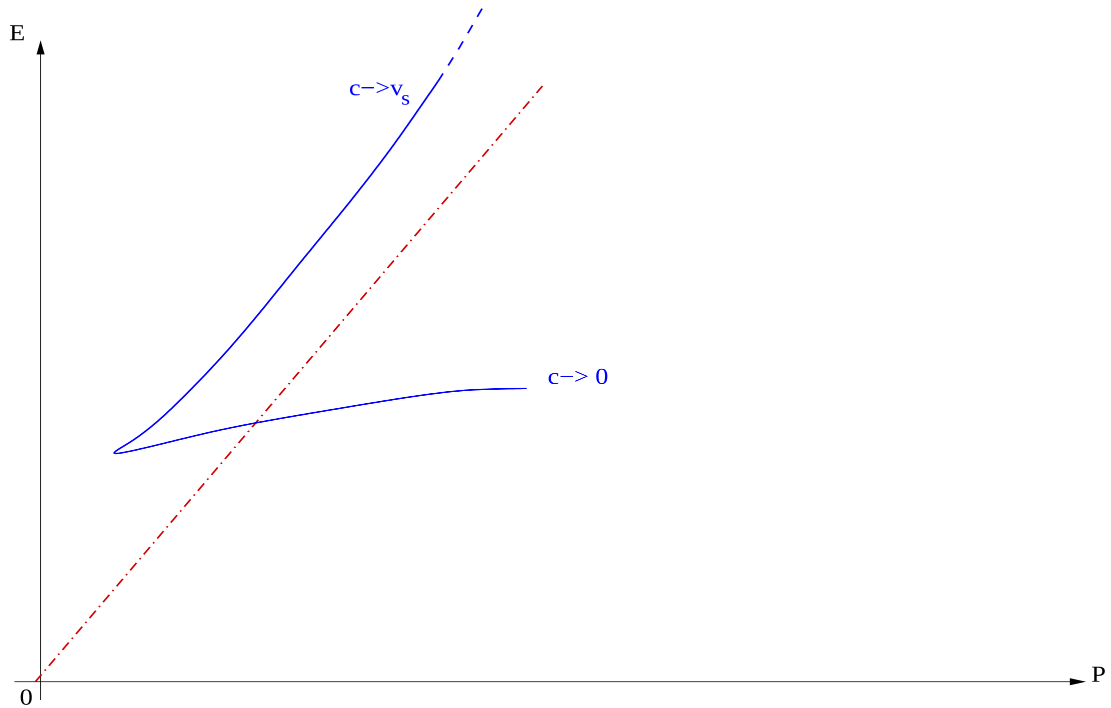
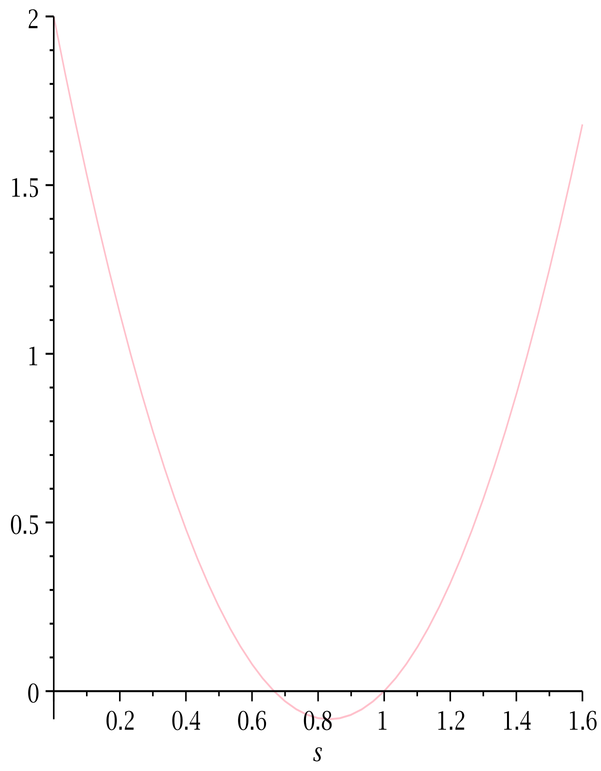
Ex4 :
$$f(r) = (r-1) - \frac{3}{2}(r-1)^2 + 2(r-1)^3 - \frac{5}{2}(r-1)^4$$
$$+3(r-1)^5 - \frac{7}{2}(r-1)^6 + 4(r-1)^7 \qquad (\Gamma > 0\ )$$



Other Energy/Momentum diagram in dimension $d = 1$

Actually, when $d = 1$, the Energy/Momentum diagrams can be curious

Ex5 : $$f(r) = (r-1) + 3(r-1)^2 \qquad (\Gamma = 24 > 0\ )$$



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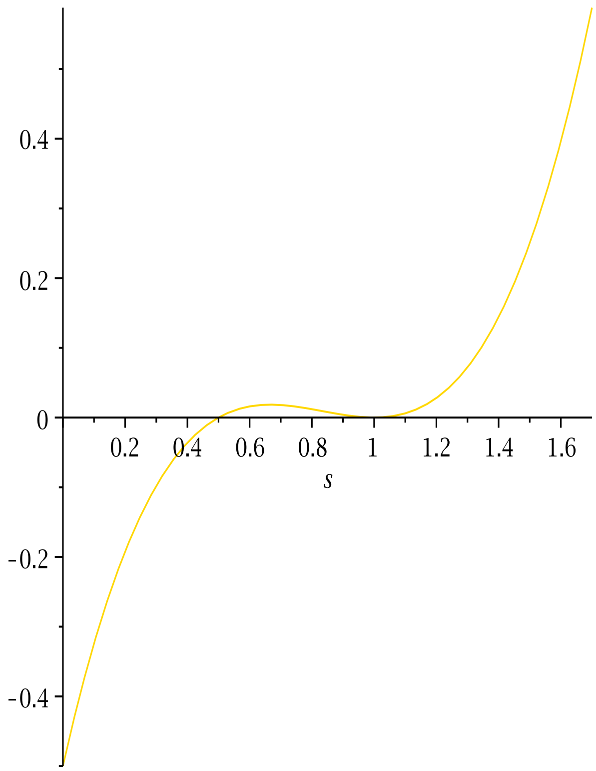

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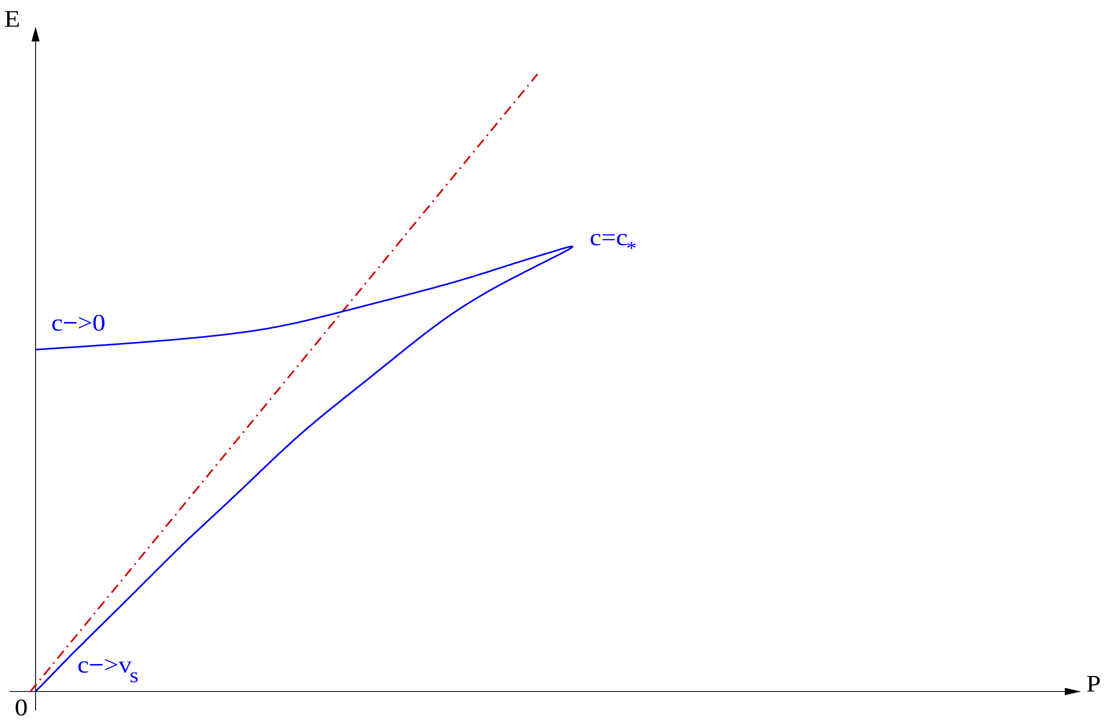

Energy/Momentum diagram in dimension $d = 2$

What about dimension $d = 2$ when $\Gamma = 0$?

With the previous ansatz,

$$E(u_\varepsilon) \simeq \mathfrak{c}_s P(u_\varepsilon) \simeq \varepsilon^{-1} \gg 1.$$

$\longrightarrow$ Does there exist rarefaction waves with high energy ?

$\longrightarrow$ Does there exist solutions with small energy (and momentum) ?

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Works in progress:

- numerical simulation in $d = 2$.
- error bounds for the (KdV)/(KP-I) asymptotic regime.
- justification of the time dependent cubic (KP-I) equation.


"That's all Folks!"